EXTREMES OF SOLUTIONS OF STOCHASTIC EQUATIONS

Panel Topic E on Uncertainty Quantification

(Oral presentation is not desired)

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Motivation: Stochastic equations, i.e., equations with random coefficients and/or end conditions, are used to describe a broad range of problems in science and engineering. The solutions of these equations are random functions of space and/or time whose probability laws need to be determined. Moments and distributions provide useful global information on these solutions that can be used to characterize, e.g., the apparent conductivity of material specimens whose small scale conductivity varies randomly in space. However, they provide limited if any information on the sample properties of these solutions which are needed to characterize, e.g., maxima of stress/strain fields in random heterogeneous microstructures, and capture changes in material properties subjected to aggressive environments such those in fusion reactors.

Most practical methods for solving stochastic equations provide global information on the solutions of these equations, e.g., the stochastic Galerkin and collocation methods [1, 2]. Monte Carlo simulation is the only general method which can be used to construct estimates for both global and sample properties of the solutions of stochastic equations. However, the method is not feasible if the calculation of a single solution sample is computationally intensive and/or the number of required samples is large. For example, Monte Carlo estimates of extremes of solutions of stochastic equations are not feasible since they require large sets of solution samples. Alternative approaches are needed for this class of stochastic problems.

Approach: It is proposed to used concepts of the extreme value theory (EVT) to construct estimates of the distributions of extremes of solutions of stochastic equations from relatively small number of samples of these solutions [4]. Solution samples can be obtained by Monte Carlo simulation or surrogate models of the type introduced in [3].

Let $\{\Sigma_{ij}(x)\}$, $x \in D$, $i, j = 1, 2$, denote the stress tensor field in a thin rectangular plate subjected to uniform deterministic tension at two opposite ends. The stresses $\{\Sigma_{ij}(x)\}$ are real-valued random fields since the stiffness tensor varies randomly in the plate domain D. Solutions of the stochastic equations defining the stress field have been obtained for 1000 independent samples of the stiffness tensor. They constitute 1000 independent samples of the tensor-valued stress random field $\{\Sigma_{ij}(x)\}$. A sample of this field is shown in Fig. 1. Our objective is to estimate the upper tail of, e.g., the probability $p_f(\sigma_{\rm cr}) = P\big(\max_{x\in D} \|\Sigma(x)\| > \sigma_{\rm cr}\big)$ from samples of the random variable $\max_{x\in D} \|\Sigma(x)\|$, where $\Sigma(x)$ is a square matrix with entries $\{\Sigma_{ij}(x)\}$, $i, j = 1, 2$. The available sample size is insufficient to estimate probabilities $p_f(\sigma_{\rm cr})$ of orders smaller than 10^{-2}. We propose to use concepts of the EVT to construct estimates of $p_f(\sigma_{\rm cr})$ of orders much smaller than 10^{-2}, which can be viewed as extensions of the empirical estimates of $p_f(\sigma_{\rm cr})$ beyond data.

The stars in Fig. 2 are the empirical estimates of $p_f(\sigma_{\rm cr})$ obtained from the entire set of 1000 samples of $\max_{x\in D} \|\Sigma(x)\|$. The heavy solid line is the EVT estimate of $p_f(\sigma_{\rm cr})$ based

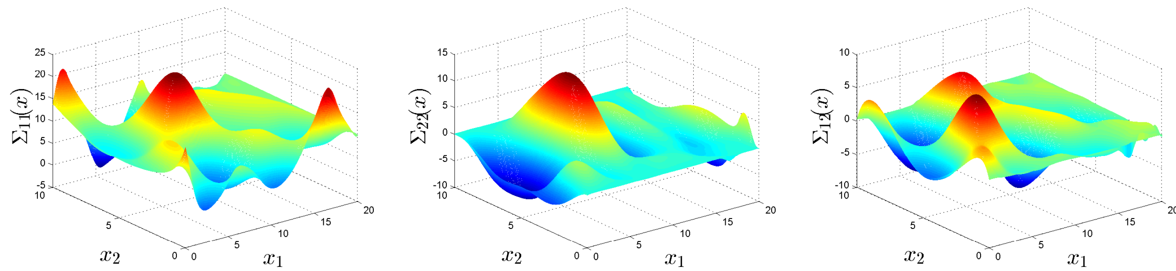

Figure 1: Samples of $\Sigma_{11}(x)$ (left panel), $\Sigma_{22}(x)$ (middle panel), and $\Sigma_{12}(x)$ (right panel)

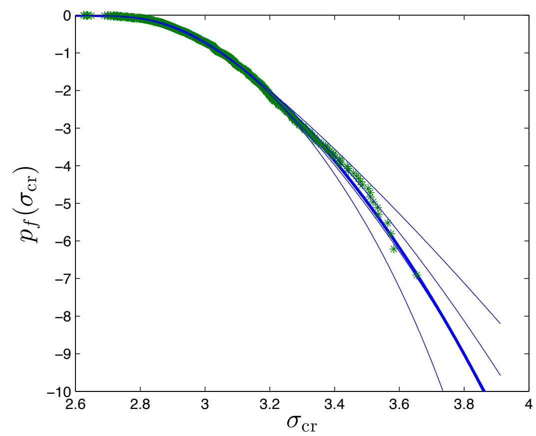

Figure 2: Empirical and GEV estimates of $p_f(\sigma_{\text{cr}})$ (logarithmic scale)

on all 1000 independent samples of $\max_{x\in D}\|\Sigma(x)\|$. It follows closely the empirical estimate of $p_f(\sigma_{\text{cr}})$. The thin solid lines are EVT estimates of $p_f(\sigma_{\text{cr}})$ based on distinct subsets of the samples of $\max_{x\in D}\|\Sigma(x)\|$ with size 250. They are consistent with the EVT estimates of $p_f(\sigma_{\text{cr}})$ based on all samples of $\max_{x\in D}\|\Sigma(x)\|$. All EVT estimates extend beyond data.

This numerical experiment suggests that the EVT can be applied to estimate the distributions of extremes of solutions of stochastic equations at probabilities levels beyond those of empirical estimates and that the EVT estimates are stable even when based on relatively small data sets. The latter feature is essential for realistic applications because the set of solution samples is usually small in realistic applications.

Impact: The EVT-based approach constitutes a new direction in the solution of stochastic equations which has notable features. It is (*i*) conceptually simple, (*ii*) nonintrusive in the sense that its implementation uses existing software for solving deterministic equations, (*iii*) general, i.e., it delivers global solution statistics, as most of the current methods, but also sample properties of these solutions, e.g., spatial and/or temporal extremes, and (*iv*) converges to the exact solution.

References

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