Break the curse of high dimensionality and low accuracy: Efficient and high order numerical methods for nonlinear filtering problems

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1 Motivation and Objective

The nonlinear filtering problem estimates the uncertainty state, such as the trajectory of a flying object, of a stochastic dynamical system from noisy observations. More precisely, the standard nonlinear filtering problem considers an unobservable state process $X = \{X_t : t \geq 0\}$, and an associated observation process $Y = \{Y_t : t \geq 0\}$ whose values are a function of X after corrupted by noise. The goal of nonlinear filtering theory is to obtain the best estimate of the uncertain state X_t of the nonlinear stochastic system from $Y_{1:t} := \{Y_s : 0 \leq s \leq t\}$, the observations of the system up until time t.

Examples of nonlinear filtering problems arise in biology, signal processing, image processing and target tracking for military purposes. As one of most powerful tools of data assimilation and uncertainty quantification, nonlinear filtering can also be very useful for magnetic fusion research. One such instance is the study of Coulomb collisions in a diluted plasma. Since it is not possible to completely determine the exact distribution of the gyro tropic beam particle injected to a set of electron particles, nonlinear filtering can help accurately simulate this distribution based partial observations.

While extensive approaches are available for the linear filtering problem, the nonlinear filtering problem, which appears in most of real world applications, remains considerably difficult to solve. When analyzed on a computer, such problems typically involve the input of a very large set of observational data; accurate modeling and analysis of the stochastic aspects of the problem require calculations in very high dimensions. Furthermore, the white noise perturbation of the dynamical system results in low regularity of the state variable, which makes it very difficult to construct high order numerical methods for continuous nonlinear filtering problems. Together, these facts make the accurate solution of nonlinear filtering problems an extraordinarily computation intensive task.

The emerging availability of exascale computing power offers a golden opportunity to develop rapid and efficient solution methods for nonlinear filtering problems of practical importance. In addition, the recent advances in UQ research and numerical solutions of stochastic partial differential equations also make it the right time to attack the difficulties of solving nonlinear filtering problems. In this proposed research we propose to construct two classes of novel numerical algorithms for the accurate approximation of nonlinear filtering problems. The first class is a complete renovation of the existing filter method called Zakai filter which consists of solving a class of high dimensional stochastic partial differential equations. Our new ideas of solving the Zakai filter include adaptive selection of the solution domain at each time step, which will result in significant reduction in computing cost, and a split-up numerical scheme on sparse grid. The second class is a completely new filtering method where a system of forward and backwardly doubly stochastic differential equations (FBDSDEs) is used to solve the nonlinear filtering problem. We name this new method FBDSDE filter. The most significant feature of FBDSDE filter is its high convergence rate that cannot be achieved by any existing nonlinear fitter method. Our research will consists of both rigorous mathematical analysis to justify the proposed numerical methodology and extensive numerical experiments to demonstrate the validity and efficiency of the proposed numerical algorithms.

2 Novel PDF methods for nonlinear filtering problems

2.1 Main challenges of the existing filtering methods The prevailing methods of solving nonlinear filtering problems in both literature and practice are extended and unscented Kalman filter, particle filter, and Zakai filter. While each of the method has its own advantages and disadvantages, they share two common challenges.

• **High dimensionality** In general, the dimensionality of the state process X is high. For Zakai filter, this means that one must solve a high dimensional stochastic partial differential equation whose solution is difficult to obtain because of the curse of dimensionality. In essence particle filters attempt to approximate the full probability density function of the state process, will be substantially more difficult to apply in high dimensions. Thus particle filters suffer the same curse of dimensionality that afflicts most approaches for the nonlinear filtering problem. Though the Kalman filter is least affected by high dimensionality, the cost of evaluating of high dimensional covariance matrices also increases dramatically as the dimension of the state process increases.

• **Low accuracy** For Zakai filter, the white noise perturbation results in the low regularity of the solution of the corresponding Zakai equation, which makes it extremely difficult to construct numerical solutions with high accuracy. In addition, the unbounded physical domain of the Zakai equation also makes it very difficult to find its numerical solution efficiently. As a sequential Monte Carlo method, particle filter suffers slow convergence as all Monte Carlo methods. The extended Kalman filter (EKF) is essentially the linearization of the nonlinear dynamical system and its results will be very poor when the dynamics systems is significantly nonlinear or the noise intensities are high.

2.2 New ideas of attacking the algorithm difficulties The goal of this proposal is to overcome the aforementioned difficulties and produce efficient and accurate numerical algorithms for nonlinear filtering problems. We will develop two classes of high order numerical algorithms: the first solves the Zakai equation directly, using the split-up scheme on a sparse grid and adaptively constructed finite domains; the second solves an equivalent class of backward stochastic differential equations.

• In the first class of algorithms, we overcome the difficulties associated with the unbounded domain by using the importance sampling method to construct bounded domains for the Zakai equation with very small number of sample points.

• The first class of algorithms will overcome the high dimensionality using a class of split-up finite different schemes on a sparse grid, reducing the order of the degrees of freedom from $O(n^d)$ to $(n \ln^{d-1} n)$.

• The second class of algorithms will overcome the problem of low accuracy, using a class of high order numerical methods for the BDSDE. A special class of Itô-Taylor formulas will provide a high order solution to the system of stochastic differential equations. This approach, which solves a system of stochastic *ordinary* differential equations, is fundamentally different and more powerful than existing methods.

• The second class of algorithms will overcome the curse of dimensionality through the introduction of mesh free approximations in spatial domain. This will be done through the importance sampling method similar to the one used to construct adaptieve domains in the first class of algorithms.