Convergence Properties of Particle-In-Cell Methods

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May 4, 2015

The Problem of Noise

The numerical solution of the high dimensional Vlasov equation is usually performed by particle-in-cell (PIC) methods. However, due to *numerical noise*, it is challenging to use PIC methods to get a precise description of the distribution function in phase space. The convergence of PIC for Vlasov-Poisson problems was described in Cottet and Raviart 1984, and extended in Wang, Miller and Colella in SIAM 2011. Starting with vortex methods, convergence can be seen to depend on particle overlap and regularization.

For a second-order method the error is measured as the difference between the approximated and the exact density function

$$\begin{aligned} e_t(x,t) &= |\rho(x,t) - \tilde{\rho}(x,t)| \\ &= |\rho(x,t) - \sum_i q_i \delta_{\Delta x}(x - \tilde{X}_i(t))| \\ &\le \underbrace{\left|\rho(x,t) - \sum_i q_i \delta_{\Delta x}(x - X_i(t))\right|}_{\textbf{consistency error}:e_c(x,t)} \\ &\quad + \underbrace{\left|\sum_i q_i \delta_{\Delta x}(x - X_i(t)) - \sum_i q_i \delta_{\Delta x}(x - \tilde{X}_i(t))\right|}_{\textbf{stability error}:e_s(x,t)} \end{aligned}$$

$$|\rho(x,t) - \tilde{\rho}(x,t)| \le c(t)(\underbrace{\Delta x^2 + \Delta x^2\left(\frac{h_x}{\Delta x}\right)^2}_{e_c(x,t)} + \underbrace{(\Delta x^2 + \Delta x^2\left(\frac{h_x}{\Delta x}\right)^2 + \Delta x^2)(\exp(a(\Delta x, h_x, t)t) - 1))}_{e_s(x,t)}.$$

$$a(\Delta x, h_x, t) \propto \left|\left|\frac{\partial E}{\partial x}\right|\right|_{L^\infty}$$

Convergence = Consistency + Stability. To have a convergent method you require overlapping: $\frac{h_x}{\Delta x} \le 1$ and regularization: $\exp(a(\Delta x, h_x, t)t) \approx 2$. There is no requirement for a plasma model to have a decaying electric field gradient, so this term only grows. The exponential regularization error

term is what commonly manifests in simulations as *numerical noise*. This exponential error growth makes long-time simulations complicated. There are mechanisms to control plasma particle noise that are used in vortex methods and smooth particle hydrodynamics and shown in [Wang, Miller, Colella 2011,2012].

Many implementations of PIC for fusion plasmas rely on implicit regularization that comes about due to the effect of field transfers to and from the grid. We do not think that is adequate. Stability and Consistency error should be addressed. Consistency tells us particle counts should scale grid resolution. When it comes to PIC modeling, the particles-per-cell is not a free parameter that can be fit to the computing platform. PIC codes are subject to convergence studies just like Eulerian models and convergence results should be part of any PIC modeling result. That is how verification is done.